Midterm Exam - Introduction to Symplectic Geometry
M. Math II

23 February, 2025

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ____________________

Roll Number: ____________________

1. (10 points) Let (V, ω) be a finite-dimensional symplectic vector space. Show that there is an $n \in \mathbb{N}$ and a basis $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ of V such that

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \omega(u_j, v_k) = \delta_{jk}$$

for $1 \leq j, k \leq n$. Total for Question 1: 10

2. (10 points) Let (V, ω) be a symplectic vector space over $\mathbb{R}$. Let W be a subspace of V and let $W^{\perp} := \{w \in V \mid \omega(v, w) = 0 \; \forall \; v \in V\}$. Show that ω induces a symplectic form on the quotient space $W/(W \cap W^{\perp})$. Total for Question 2: 10

3. (a) (10 points) Let (V, ω) be a symplectic vector space over $\mathbb{R}$. Show that a linear map $\Psi : V \to V$ is a symplectomorphism if and only if its graph

$$\{(v, \Psi(v)) : v \in V\}$$

is a Lagrangian subspace of $(V \oplus V, -\omega \oplus \omega)$.

(b) (15 points) Let $W \subset \mathbb{R}^{2n}$ be a subspace of the form

$$W = \{(u, Au) : u \in \mathbb{R}^n\}$$

with $A \in M_n(\mathbb{R})$. Prove that W is Lagrangian if and only if A is symmetric.

Total for Question 3: 25

4. (a) (15 points) Prove that $W \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if and only if there exists a unitary matrix $U = X + iY \in U(n)$ (X, Y are matrices containing entrywise real part, imaginary part of U, respectively) such that

$$W = W_U := \left\{ \begin{pmatrix} Xu \\ Yu \end{pmatrix} : u \in \mathbb{R}^n \right\}.$$

(b) (10 points) Let $U, V \in U(n)$. Prove that $W_U = W_V$ if and only if $UU^T = VV^T = I_n$.

Total for Question 4: 25

Let (V, ω) be a symplectic vector space over $\mathbb{R}$. A complex structure J is said to be ω-compatible, that is, $\omega(Jx, Jy) = \omega(x, y)$ for all $x, y \in V$. An ω-compatible complex structure J on V is said to be positive if $g(x, y) := \omega(x, Jy)$ defines a positive-definite symmetric bilinear form.

5. (a) (15 points) Prove that a complex structure J is ω-compatible if and only if there exists a standard symplectic basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ such that $v_j = Ju_j$ for $1 \leq j \leq n$.

(b) (15 points) Prove that the space of ω-compatible positive complex structures is non-empty.

Total for Question 5: 30

6. (10 points) Let (V, ω) be a real symplectic space and J a positive compatible complex structure. Show that

$$U(V, J) = Sp(V) \cap O(V, g).$$

Total for Question 6: 10